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PERIODICITY OF d-CLUSTER-TILTED ALGEBRAS

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Abstract. It is well-known that any maximal Cohen-Macaulay module over a hypersurface has a periodic free resolution of period 2. Auslander, Reiten [4] and Buchweitz [7] have used this periodicity to explain the existence of periodic projective resolutions over certain finite-dimensional algebras which arise as stable endomorphism rings of Cohen-Macaulay modules. These algebras are in fact periodic, meaning that they have periodic projective resolutions as bimodules and thus periodic Hochschild cohomology as well. The goal of this article is to generalize this construction of periodic algebras to the context of Iyama's higher AR-theory. We let $\mathcal{C}$ be a maximal $(d-1)$-orthogonal subcategory of an exact Frobenius category $\mathcal{B}$, and start by studying the projective resolutions of finitely presented functors on the stable category $\underline{\mathcal{C}}$, over both $\underline{\mathcal{C}}$ and $\mathcal{C}$. Under the assumption that $\underline{\mathcal{C}}$ is fixed by Ω^d, we show that Ω^d induces the $(2+d)^{th}$ syzygy on mod-$\underline{\mathcal{C}}$. If $\mathcal{C}$ has finite type, i.e., if $\mathcal{C} = \mathrm{add}(T)$ for a d-cluster tilting object T, then we show that the stable endomorphism ring of T has a quasi-periodic resolution over its enveloping algebra. Moreover, this resolution will be periodic if some power of Ω^d is isomorphic to the identity on $\underline{\mathcal{C}}$. It follows, in particular, that 2-C.Y.-tilted algebras arising as stable endomorphism rings of Cohen-Macaulay modules over curve singularities, as in the work of Burban, Iyama, Keller and Reiten [8], have periodic bimodule resolutions of period 4.

1. Introduction

In this article we describe a new way of constructing finite-dimensional endomorphism algebras with periodic Hochschild (co)homology. In fact, we show that the endomorphism rings we consider are *periodic* in the sense that they have periodic projective resolutions over their enveloping algebras; i.e., $\Omega^n_{A^e}(A) \cong A$ as bimodules for some $n > 0$. Among the most notable examples of finite-dimensional algebras with this property are the preprojective algebras of Dynkin graphs, which all have period 6. This interesting fact was first proved by Ringel and Schofield through a calculation of the minimal projective bimodule resolutions of such algebras. Later, Auslander and Reiten [4] gave an elegant functorial argument for this periodicity, making use of the fact that these preprojective algebras can be realized as stable endomorphism rings of Cohen-Macaulay modules (in fact, as stable Auslander algebras) over 2-dimensional simple hypersurface singularities. Actually, their arguments establish a slightly weaker version of this periodicity, showing only that the sixth power of the syzygy functor is the identity. Motivated by these results, Buchweitz [7] develops the functor category arguments of Auslander and Reiten to deduce the (full) periodicity of the preprojective algebras of Dynkin graphs from the isomorphisms $\Omega^2 \cong Id$ in the corresponding stable categories of CM-modules. More generally, his work shows how periodic algebras can arise as stable Auslander algebras of finite-type categories, and in particular as stable endomorphism rings of Ω-periodic modules.

Iyama has recently developed higher-dimensional analogues of much of the classical Auslander-Reiten theory, including a theory of higher Auslander algebras [16, 17]. Thus it is natural to look for generalizations of Auslander, Reiten and Buchweitz's work on periodicity to this setting. One clue is already provided by recent work of Burban, Iyama, Keller and Reiten [8], showing that symmetric algebras with τ-period 2 can be obtained as endomorphism rings of certain Cohen-Macaulay modules over 1-dimensional hypersurface singularities. Among the algebras they realize in this way are several algebras of quaternion type, which Erdmann and Skowronski have shown are periodic of period 4 [13]. As Erdmann and Skowronski's result is obtained by computing minimal projective resolutions over enveloping algebras, our motivation is parallel to Buchweitz's in [7]. That is, we aim to generalize Buchweitz's results to explain how the 2-periodicity of the syzygy functor in the category of CM-modules implies the 4-periodicity of the bimodule resolutions for the appropriate endomorphism rings.

Key words and phrases. periodic algebra, maximal orthogonal subcategory, cluster-tilting object, higher Auslander algebra.

It turns out that we can obtain periodic algebras more generally as endomorphism rings of periodic d-cluster-tilting objects in a triangulated category. These d-cluster-tilting objects are in fact the objects T for which $\mathrm{add}(T)$ satisfies Iyama's definition of a maximal $(d-1)$-orthogonal subcategory. Hence our results are indeed analogues of Buchweitz's for Iyama's higher Auslander-Reiten theory. We summarize our main results in the theorem below, where $\mathcal{B}$ denotes an exact Frobenius category with a Hom-finite stable category $\underline{\mathcal{B}}$.

Theorem 1.1. *Let T be a d-cluster tilting object in $\mathcal{B}$ (with $d \geq 1$) such that $\Omega^d T \cong T$ in $\underline{\mathcal{B}}$, and set $\Lambda = \mathrm{End}_{\mathcal{B}}(T)$ and $\Gamma = \underline{\mathrm{End}}_{\mathcal{B}}(T)$. Then*

(1) $\mathrm{Tor}_i^{\Lambda}(-,\Gamma) = 0$ *on mod-*Γ *for all* $i \neq 0, d+1$.
(2) $\Omega^{d+2}_{\Gamma^e}(\Gamma) \cong \mathrm{Tor}^{\Lambda}_{d+1}(\Gamma,\Gamma) \cong \underline{\mathcal{B}}(T, \Omega^d T)$ *is an invertible* (Γ,Γ)*-bimodule.*
(3) *If* Ω^d *has order* r *as a functor on* $\underline{\mathrm{add}(T)}$, *then* Γ *is periodic with period dividing* $(d+2)r$.

For $d = 1$, the same conclusions were obtained by Buchweitz [7] under the assumption (needed for (2) and (3)) that Λ has Hochschild dimension $d+1 = 2$. He then applies it to an additive generator T of the finite-type category $\mathcal{B} = CM(R)$ for a simple hypersurface singularity R of dimension 2 in order to deduce the periodicity of the preprojective algebras of Dynkin type. For $d = 2$, we can again take $\mathcal{B} = CM(R)$ for an odd-dimensional isolated Gorenstein hypersurface (see [21] for instance). Since Eisenbud's matrix factorization theorem [11] implies that $\Omega^2 \cong Id$ on $\underline{\mathcal{B}}$ in this case, any 2-cluster-tilting object in $\underline{\mathcal{B}}$ is automatically 2-periodic and thus has a stable endomorphism algebra which is periodic of period 4. Existence of 2-cluster-tilting objects in this setting has been studied by Burban, Iyama, Keller and Reiten [8]. We will discuss this and other potential applications further in the final section.

We typically work with right modules, unless noted otherwise. In this case morphisms are written on the left and composed from right to left. We also follow this convention for morphisms in abstract categories, as well as for paths in quivers.

2. Functors on maximal orthogonal subcategories

Throughout this article, we let $\mathcal{B}$ be an exact Krull-Schmidt, Frobenius k-category for a field k. In particular $\mathcal{B}$ has enough projectives and enough injectives and these coincide. We denote the stable category by $\underline{\mathcal{B}}$, which is a triangulated category with the cosyzygy functor Ω^{-1} as its suspension [15]. In $\underline{\mathcal{B}}$ we will often write $X[i]$ for the i^{th} suspension $\Omega^{-i}X$ of X. We write $\underline{f}$ for the residue class in $\underline{\mathcal{B}}$ of a map f in $\mathcal{B}$. We further assume that all the Hom-spaces $\underline{\mathcal{B}}(X,Y)$ are finite-dimensional over k. Typically, we have in mind for $\mathcal{B}$ either (an exact subcategory of) mod-A for a finite-dimensional self-injective k-algebra A or else the category $CM(R)$ of maximal Cohen-Macaulay modules over an isolated Gorenstein singularity R (containing k).

We assume that $\mathcal{C}$ is a functorially finite, maximal $(d-1)$-orthogonal subcategory of $\mathcal{B}$ for some $d \geq 1$. Recall that the latter condition means

$$\mathcal{C} \quad = \quad \{X \in \mathcal{B} \mid \mathrm{Ext}^i_{\mathcal{B}}(X,\mathcal{C}) = 0, \forall\ 1 \leq i < d\} = \{Y \in \mathcal{B} \mid \mathrm{Ext}^i_{\mathcal{B}}(\mathcal{C},Y) = 0, \forall\ 1 \leq i < d\}. \tag{2.1}$$

In particular, $\mathcal{C}$ must contain all the projectives in $\mathcal{B}$, and we have $\underline{\mathcal{B}}(\mathcal{C},\mathcal{C}[i]) = 0$ for all $1 \leq i < d$. If $\mathcal{C} = \mathrm{add}(T)$ for an object $T \in \mathcal{B}$, then we say that T is a d-cluster tilting object (in $\mathcal{B}$ or in $\underline{\mathcal{B}}$). Notice that in this case $\mathcal{C}$ will automatically be functorially finite. We point out that for $d = 1$ this definition forces $\mathcal{C} = \mathcal{B}$, which brings us back essentially to the setting considered by Auslander and Reiten in [1] and Buchweitz in [7]. With $\mathcal{C}$ and d fixed we also define subcategories

$$\mathcal{E}_j \quad = \quad \{X \in \mathcal{B} \mid \underline{\mathcal{B}}(\mathcal{C}, X[i]) = 0 \text{ for } 1 \leq i \leq d-1 \text{ and } i \neq j\} \tag{2.2}$$

for each $1 \leq j \leq d$. Notice that $\mathcal{E}_d = \mathcal{C}$ and $\mathcal{C} \cup \mathcal{C}[1] \subseteq \mathcal{E}_{d-1}$. If $d = 2$ then $\mathcal{E}_1 = \mathcal{B}$.

Our main results require an additional stronger vanishing condition on $\mathcal{C}$. Fortunately, it turns out to be equivalent to a more natural (and more easily checked) periodicity condition, as we now verify.

Lemma 2.1. *For $\mathcal{C}$ and $\mathcal{B}$ as above, the following are equivalent.*

(1) $\underline{\mathcal{B}}(\mathcal{C},\mathcal{C}[i]) = 0$ *for all* i *with* $-d < i \leq -1$.
(2) $\underline{\mathcal{C}}[d] = \underline{\mathcal{C}}$; *that is,* $\Omega^d C \in \mathcal{C}$ *for each* $C \in \mathcal{C}$.

Proof. For $X \in \mathcal{C}$, notice that $X[d] \in \mathcal{C}$ if and only if $\underline{\mathcal{B}}(X[d], \mathcal{C}[i]) = 0$ for $1 \leq i < d$, which is equivalent to $\underline{\mathcal{B}}(X, \mathcal{C}[j]) = 0$ for $-d < j \leq -1$. $\square$

We will often assume that $\mathcal{C}$ satisfies the two equivalent conditions of the above lemma. Note that these are automatic for $d = 1$ and $\mathcal{C} = \mathcal{B}$. In case $\underline{\mathcal{B}}$ has Serre duality $\underline{\mathcal{B}}(X, SY) \cong D\underline{\mathcal{B}}(Y, X)$ for an auto-equivalence S of $\underline{\mathcal{B}}$, then the above conditions are easily seen to be equivalent to $S(\mathcal{C}) = \mathcal{C}$.

The following lemma is useful for obtaining exact sequences in $\mathcal{B}$, which may fail to be an abelian category.

Lemma 2.2. *For any map $f : X \to Y$ in $\mathcal{B}$, there exists an object Z and a projective P in $\mathcal{B}$ such that $0 \to Z \longrightarrow X \oplus P \xrightarrow{(f\ p)} Y \to 0$ is exact in $\mathcal{B}$.*

Proof. We can complete $\underline{f}$ to a triangle $Z \to X \to Y \to$ in $\underline{\mathcal{B}}$. Up to isomorphism, this triangle is induced by a short exact sequence $0 \to Z \longrightarrow X \oplus E(Z) \longrightarrow Y \oplus Q \to 0$ in $\mathcal{B}$, where $E(Z)$ is the injective envelope of Z and Q is projective. The pull-back of this sequence via the obvious split monomorphism $Y \to Y \oplus Q$ now has the desired form. $\square$

We use the standard notation mod-$\mathcal{C}$ and mod-$\underline{\mathcal{C}}$ for the categories of finitely presented contravariant k-linear functors from $\mathcal{C}$ and $\underline{\mathcal{C}}$, respectively, to mod-k. As we only consider functors on $\mathcal{C}$ or $\underline{\mathcal{C}}$, and never on $\mathcal{B}$, all representable functors $\mathcal{B}(-, X)$ or $\underline{\mathcal{B}}(-, X)$ are to be interpreted as restricted to $\mathcal{C}$, and we forgo writing $\mathcal{B}(-, X)|_{\mathcal{C}}$ for the restriction. Our current goal is to describe the projective resolutions of finitely presented $\underline{\mathcal{C}}$-modules over $\underline{\mathcal{C}}$ and over $\mathcal{C}$. We start with a simple but important observation that generalizes a theorem of Buan, Marsh and Reiten for 2-cluster tilting objects in cluster categories [6] (see also Corollary 6.4 in [19]).

Lemma 2.3. *Let $\mathcal{B}$ and $\mathcal{C}$ be as above, and assume $d \geq 2$.*

1. *For any $M \in$ mod-$\underline{\mathcal{C}}$, we have $M \cong \underline{\mathcal{B}}(-, X)$ for some $X \in \mathcal{E}_{d-1}$ (without projective summands).*
2. *The functor $\eta : \underline{\mathcal{B}} \longrightarrow$ mod-$\underline{\mathcal{C}}$ given by $\eta(X) = \underline{\mathcal{B}}(-, X)$ is full and dense. Moreover, the restriction of η to $\underline{\mathcal{E}_{d-1}}$ induces a category equivalence*

$$\eta : \underline{\mathcal{E}_{d-1}}/\langle \underline{\mathcal{C}}[1] \rangle \xrightarrow{\approx} \text{mod-}\underline{\mathcal{C}}.$$

In particular, if $\underline{\mathcal{B}}$ has finite type, then so does mod-$\underline{\mathcal{C}}$.

Proof. A minimal projective presentation of M in mod-$\underline{\mathcal{C}}$ has the form

$$\underline{\mathcal{B}}(-, C_1) \xrightarrow{\underline{\mathcal{B}}(-, \underline{f})} \underline{\mathcal{B}}(-, C_0) \longrightarrow M \to 0 \tag{2.3}$$

for a map $f : C_1 \to C_0$ in $\mathcal{C}$. We can complete $\underline{f}$ to a triangle $C_1 \xrightarrow{\underline{f}} C_0 \longrightarrow X \longrightarrow$ in $\underline{\mathcal{B}}$. The long-exact Hom-sequence now yields the exact sequence (using $d \geq 2$)

$$\underline{\mathcal{B}}(-, C_1) \xrightarrow{\underline{\mathcal{B}}(-, \underline{f})} \underline{\mathcal{B}}(-, C_0) \longrightarrow \underline{\mathcal{B}}(-, X) \longrightarrow \underline{\mathcal{B}}(-, C_1[1]) = 0, \tag{2.4}$$

whence $M \cong \underline{\mathcal{B}}(-, X)$. Furthermore, the exact sequences

$$0 = \underline{\mathcal{B}}(-, C_0[i]) \longrightarrow \underline{\mathcal{B}}(-, X[i]) \longrightarrow \underline{\mathcal{B}}(-, C_1[i+1]) = 0$$

for $1 \leq i \leq d-2$ show that $X \in \mathcal{E}_{d-1}$.

It follows easily that η (even restricted to $\underline{\mathcal{E}_{d-1}}$) is full and dense, so we need only compute its kernel on $\underline{\mathcal{E}_{d-1}}$. Clearly the kernel contains the ideal $\langle \underline{\mathcal{C}[1]} \rangle$ since $\underline{\mathcal{B}}(-, C[1]) = 0$ for all $C \in \mathcal{C}$. Now let $f : X \to Y$ be a map between two objects of $\underline{\mathcal{E}_{d-1}}$ such that $\underline{\mathcal{B}}(C, f) = 0$ for all $C \in \mathcal{C}$. If we complete a right $\underline{\mathcal{C}}$-approximation $g : C_0 \to X$ to a triangle $Z \longrightarrow C_0 \longrightarrow X \to$ in $\underline{\mathcal{B}}$, then the induced long exact sequence of representable functors on $\mathcal{C}$ shows that $Z \in \mathcal{E}_d = \mathcal{C}$. As $fg = 0$ by assumption, we know that f must factor through the connecting morphism $X \to Z[1]$, whence f is in the ideal generated by $\underline{\mathcal{C}}[1]$. $\square$

Remark. Of course, the final statement fails for $d = 1$ as it is well-known that the stable Auslander algebra of a self-injective algebra of finite representation type usually has infinite representation type.

Proposition 2.4. *Let $M \in$ mod-$\underline{\mathcal{C}}$, and assume that $d \geq 2$ and $\underline{\mathcal{C}}[d] = \underline{\mathcal{C}}$.*

(1) *There is a projective presentation of M in mod-$\mathcal{C}$ of the form*

$$0 \to \mathcal{B}(-,\Omega X) \longrightarrow \mathcal{B}(-,C_1) \xrightarrow{\mathcal{B}(-,f)} \mathcal{B}(-,C_0) \longrightarrow M \to 0$$

for $C_0, C_1 \in \mathcal{C}$ and some $X \in \mathcal{B}$ with $M \cong \underline{\mathcal{B}}(-,X)$.

(2) *The above sequence induces the following projective presentation of M in mod-$\underline{\mathcal{C}}$*

$$0 \to \underline{\mathcal{B}}(-,\Omega X) \longrightarrow \underline{\mathcal{B}}(-,C_1) \xrightarrow{\underline{\mathcal{B}}(-,f)} \underline{\mathcal{B}}(-,C_0) \longrightarrow M \to 0.$$

(3) *For any $X \in \mathcal{E}_{d-1}$ we have a natural isomorphism $\Omega^2[\underline{\mathcal{B}}(-,X)] \cong \underline{\mathcal{B}}(-,\Omega X)$ in $\underline{\text{mod}}$-$\underline{\mathcal{C}}$.*

Proof. As in the preceding proof we can find $X \in \mathcal{E}_{d-1}$ with $M \cong \underline{\mathcal{B}}(-,X)$. Keeping the notation introduced there and continuing the sequence (2.3) to the left, we obtain the exact sequence

$$0 \to \underline{\mathcal{B}}(-,\Omega X) \longrightarrow \underline{\mathcal{B}}(-,C_1) \xrightarrow{\underline{\mathcal{B}}(-,f)} \underline{\mathcal{B}}(-,C_0) \longrightarrow M \to 0$$

as $\underline{\mathcal{B}}(-,C_0[-1]) = 0$. This sequence establishes (2) and also induces the isomorphism in (3), which can be seen to be natural in $X \in \mathcal{E}_{d-1}$. Using Lemma 2.2 we now lift the triangle $C_1 \xrightarrow{f} C_0 \longrightarrow X \longrightarrow$ to a short exact sequence $0 \to C_1 \longrightarrow C_0 \oplus P \xrightarrow{(f\ p)} X \to 0$ in $\mathcal{B}$ with P projective. It follows that

$$0 \to \mathcal{B}(-,C_1) \longrightarrow \mathcal{B}(-,C_0 \oplus P) \longrightarrow \mathcal{B}(-,X) \to 0 \tag{2.5}$$

is a projective resolution of $\mathcal{B}(-,X)$ in mod-$\mathcal{C}$. The short exact sequence $0 \to \Omega X \longrightarrow P_X \xrightarrow{\pi_X} X \to 0$ yields the exact sequence

$$0 \to \mathcal{B}(-,\Omega X) \longrightarrow \mathcal{B}(-,P_X) \xrightarrow{\mathcal{B}(-,\pi_X)} \mathcal{B}(-,X) \longrightarrow \underline{\mathcal{B}}(-,X) \to 0 \tag{2.6}$$

in mod-$\mathcal{C}$. Writing $\mathcal{P}(-,X)$ for the image of $\mathcal{B}(-,\pi_X)$, we can obtain the projective presentation of $M \cong \underline{\mathcal{B}}(-,X)$ as the mapping cone of the map from the sequence

$$0 \to \mathcal{B}(-,\Omega X) \longrightarrow \mathcal{B}(-,P_X) \longrightarrow \mathcal{P}(-,X) \to 0$$

to the sequence (2.5) which is induced by the inclusion $\mathcal{P}(-,X) \to \mathcal{B}(-,X)$. Renaming $C_0 := C_0 \oplus P$ and $C_1 := C_1 \oplus P_X$ we see that this mapping cone has the desired form as in (1). □

Remark. If $d = 1$ and $\mathcal{C} = \mathcal{B}$, then the entire projective resolution of any $M = \underline{\mathcal{B}}(-,X)$ in mod-$\mathcal{C}$ has the form (2.6) (cf. [1, 2]), which is an instance of the presentation in part (1) of the proposition. In this case, parts (2) and (3) are not really applicable since $M = \underline{\mathcal{B}}(-,X)$ will be projective in mod-$\underline{\mathcal{C}}$, but part (1) yields natural isomorphisms $\Omega^2[\underline{\mathcal{B}}(-,X)] \cong \mathcal{B}(-,\Omega X)$ in mod-$\mathcal{C}$ for any $X \in \mathcal{B}$.

We now describe the remaining terms of these projective resolutions for arbitrary $d \geq 2$. Unfortunately, $X \in \mathcal{E}_{d-1}$ usually does not imply $\Omega X \in \mathcal{E}_{d-1}$, and hence we cannot simply repeat the same construction to obtain a projective resolution in mod-$\underline{\mathcal{C}}$. Instead, we find that we can iterate the construction of the first $d+1$ terms of this resolution, yielding a natural quasi-periodic resolution. Before stating the theorem we explain some of our notation. Corresponding to the natural functor $\mathcal{C} \to \underline{\mathcal{C}}$, we have an inclusion functor mod-$\underline{\mathcal{C}} \to$ mod-$\mathcal{C}$ which has a right-exact left adjoint defined by $\mathcal{B}(-,X) \mapsto \underline{\mathcal{B}}(-,X)$ for each $X \in \mathcal{B}$. We interpret this functor, which takes $\mathcal{C}$-modules to $\underline{\mathcal{C}}$-modules, as tensoring with $\underline{\mathcal{C}}$ over $\mathcal{C}$, and we write $\operatorname{Tor}_*^{\mathcal{C}}(-,\underline{\mathcal{C}})$ for its left derived functors.

Theorem 2.5. *Let $\mathcal{C}$ be a maximal $(d-1)$-orthogonal subcategory of $\mathcal{B}$ with $\underline{\mathcal{C}}[d] = \underline{\mathcal{C}}$ and $d \geq 2$, and let $M \in$ mod-$\underline{\mathcal{C}}$.*

(1) *M has a projective resolution in mod-$\mathcal{C}$ of the form*

$$0 \to \mathcal{B}(-,C_{d+1}) \to \cdots \to \mathcal{B}(-,C_1) \longrightarrow \mathcal{B}(-,C_0) \longrightarrow M \to 0$$

with each $C_i \in \mathcal{C}$.

(2) *The induced sequence of functors on $\underline{\mathcal{C}}$*

$$0 \to \operatorname{Tor}_{d+1}^{\mathcal{C}}(M,\underline{\mathcal{C}}) \longrightarrow \underline{\mathcal{B}}(-,C_{d+1}) \to \cdots \to \underline{\mathcal{B}}(-,C_0) \longrightarrow M \to 0$$

is exact, and hence yields the first $d+2$ terms of a projective resolution for M in mod-$\underline{\mathcal{C}}$.

(3) $\operatorname{Tor}_i^{\mathcal{C}}(M,\underline{\mathcal{C}}) = 0$ *for all $i \neq 0, d+1$.*

(4) *We have isomorphisms* $\mathrm{Tor}^{\mathcal{C}}_{d+1}(M, \underline{\mathcal{C}}) \cong \Omega^{d+2}_{\underline{\mathcal{C}}}(M)$ *in* $\underline{\text{mod}}$-$\underline{\mathcal{C}}$ *which are natural in* M.

(5) *For any* $X \in \mathcal{E}_{d-1}$, *we have a natural isomorphism* $\Omega^{d+2}[\underline{\mathcal{B}}(-, X)] \cong \underline{\mathcal{B}}(-, \Omega^d X)$ *in* $\underline{\text{mod}}$-$\underline{\mathcal{C}}$.

Proof. As in Proposition 2.3, there is a triangle $C_1 \to C_0 \to X \to$ in $\underline{\mathcal{B}}$ with $M \cong \underline{\mathcal{B}}(-, X)$ and $X \in \mathcal{E}_{d-1}$. Thus $\Omega X = X[-1] \in \mathcal{E}_1$. We set $L_1 := \Omega X$, and recursively define L_j for $j \geq 2$ as follows: Take a right $\underline{\mathcal{C}}$-approximation $f_j : C_j \to L_{j-1}$ and complete it to a triangle $L_j \longrightarrow C_j \xrightarrow{f_j} L_{j-1} \to$ in $\underline{\mathcal{B}}$.

We prove by induction that

(i) $L_j \in \mathcal{E}_j$ for each $1 \leq j \leq d$; and

(ii) $\underline{\mathcal{B}}(-, L_j[j-d]) \cong \underline{\mathcal{B}}(-, X[-d])$ for $1 \leq j \leq d-1$.

For $j = 1$, we have already noted that (i) holds, and (ii) is trivial. Now assume that both statements hold for some j with $1 \leq j \leq d$. We consider the exact sequences in mod-$\underline{\mathcal{C}}$ for various i

$$\underline{\mathcal{B}}(-, L_j[i-1]) \longrightarrow \underline{\mathcal{B}}(-, L_{j+1}[i]) \longrightarrow \underline{\mathcal{B}}(-, C_{j+1}[i]).$$

By hypothesis, the first term vanishes for all i with $2 \leq i \leq d$ and $i \neq j+1$; while the third term vanishes for all i with $1 \leq i \leq d-1$. We thus see that the middle term vanishes for all $i \neq j+1$ with $2 \leq i \leq d-1$. It vanishes for $i = 1$ since f_j is a right $\underline{\mathcal{C}}$-approximation, making $\underline{\mathcal{B}}(-, f_j)$ surjective. This establishes $L_{j+1} \in \mathcal{E}_{j+1}$. In particular, observe that $C_{d+1} := L_d \in \mathcal{E}_d = \mathcal{C}$. To see (ii), assume $j < d-1$ and notice that $\underline{\mathcal{B}}(-, L_{j+1}[j+1-d]) \cong \underline{\mathcal{B}}(-, L_j[j-d]) \cong \underline{\mathcal{B}}(-, X[-d])$ since $\underline{\mathcal{B}}(\mathcal{C}, C_{j+1}[i]) = 0$ for $i = j+1-d, j-d$.

For each j with $1 \leq j \leq d-2$ we now have a short exact sequence

$$0 \to \underline{\mathcal{B}}(-, L_{j+1}) \longrightarrow \underline{\mathcal{B}}(-, C_{j+1}) \longrightarrow \underline{\mathcal{B}}(-, L_j) \to 0 \tag{2.7}$$

in mod-$\underline{\mathcal{C}}$ since $\underline{\mathcal{B}}(\mathcal{C}, L_j[-1]) \cong \underline{\mathcal{B}}(\mathcal{C}[d], L_j[d-1]) \cong \underline{\mathcal{B}}(\mathcal{C}, L_j[d-1]) = 0$ and f_{j+1} is a right $\underline{\mathcal{C}}$-approximation. Splicing these sequences together yields an exact sequence

$$\underline{\mathcal{B}}(-, C_{d+1}) \to \underline{\mathcal{B}}(-, C_d) \to \cdots \to \underline{\mathcal{B}}(-, C_2) \to \underline{\mathcal{B}}(-, \Omega X) \to 0 \tag{2.8}$$

in mod-$\underline{\mathcal{C}}$, which can be viewed as the beginning of a projective resolution for $\underline{\mathcal{B}}(-, \Omega X)$. In addition, using the triangle $C_{d+1} \longrightarrow C_d \longrightarrow L_{d-1} \to$, we see that the kernel of the left-most map in this resolution is $\underline{\mathcal{B}}(-, L_{d-1}[-1]) \cong \underline{\mathcal{B}}(-, \Omega^d X)$ by (ii).

At the same time, applying Lemma 2.2 to each triangle $L_j \longrightarrow C_j \xrightarrow{f_j} L_{j-1} \to$ we obtain exact sequences $0 \to L_j \longrightarrow C_j \oplus P_j \longrightarrow L_{j-1} \to 0$ in $\mathcal{B}$ and exact sequences $0 \to \mathcal{B}(-, L_j) \longrightarrow \mathcal{B}(-, C_j \oplus P_j) \longrightarrow \mathcal{B}(-, L_{j-1}) \to 0$ in mod-$\mathcal{C}$. Splicing these together, we obtain a projective resolution for $\mathcal{B}(-, \Omega X)$ in mod-$\mathcal{C}$

$$0 \to \mathcal{B}(-, C_{d+1}) \longrightarrow \mathcal{B}(-, C_d \oplus P_d) \to \cdots \to \mathcal{B}(-, C_2 \oplus P_2) \longrightarrow \mathcal{B}(-, \Omega X) \to 0. \tag{2.9}$$

Combining this with the projective presentation in Proposition 2.4, yields the desired resolution of M. If we now apply $- \otimes_{\mathcal{C}} \underline{\mathcal{C}}$ to this resolution, the exactness of (2.8) and of $0 \to \underline{\mathcal{B}}(-, \Omega X) \longrightarrow \underline{\mathcal{B}}(-, C_1) \longrightarrow \underline{\mathcal{B}}(-, C_0) \longrightarrow M \to 0$ shows that $\mathrm{Tor}^{\mathcal{C}}_i(M, \underline{\mathcal{C}}) = 0$ for all $i \neq 0, d+1$, and $\mathrm{Tor}^{\mathcal{C}}_{d+1}(M, \underline{\mathcal{C}}) \cong \Omega^{d+2}_{\underline{\mathcal{C}}}(M)$. $\square$

If $M = \underline{\mathcal{B}}(-, C)$ for a nonprojective $C \in \mathcal{C}$, then the projective resolution in mod-$\mathcal{C}$ from the above theorem takes on an even simpler form. As in Proposition 2.4, the second syzygy of M is isomorphic to $\mathcal{B}(-, \Omega C)$. Since $\underline{\mathcal{B}}(\mathcal{C}, \Omega C) = 0$ the projective cover of ΩC will be a right $\mathcal{C}$-approximation. We thus obtain an exact sequence $0 \to \mathcal{B}(-, \Omega^2 C) \longrightarrow \mathcal{B}(-, P_2) \longrightarrow \mathcal{B}(-, \Omega C) \to 0$ in mod-$\mathcal{C}$ with P_2 projective. Repeating this construction, using $\underline{\mathcal{B}}(\mathcal{C}, \Omega^i C) = 0$ for $1 \leq i \leq d-1$, we obtain the projective resolution:

$$0 \to \mathcal{B}(-, \Omega^d C) \longrightarrow \mathcal{B}(-, P_d) \to \cdots \to \mathcal{B}(-, P_2) \longrightarrow \mathcal{B}(-, P_C) \longrightarrow \mathcal{B}(-, C) \longrightarrow \underline{\mathcal{B}}(-, C) \to 0$$

with $\Omega^d C \in \mathcal{C}$ by assumption. Passing to $\underline{\mathcal{B}}$ by factoring out the maps that factor through projectives, all terms of this projective resolution vanish except for the 0^{th} and $(d+1)^{th}$ terms. In particular, we recover the following isomorphisms

$$\mathrm{Tor}^{\mathcal{C}}_{d+1}(\underline{\mathcal{B}}(-, C), \underline{\mathcal{C}}) \cong \underline{\mathcal{B}}(-, \Omega^d C) \tag{2.10}$$

of functors on $\underline{\mathcal{C}}$, which are natural in C (note that they also follow from combining parts (4) and (5)). Thus we have isomorphisms of bifunctors on $\underline{\mathcal{C}}$

$$\mathrm{Tor}^{\mathcal{C}}_{d+1}(\underline{\mathcal{B}}(-, -), \underline{\mathcal{C}}) \cong \underline{\mathcal{B}}(-, \Omega^d(-)). \tag{2.11}$$

3. Bimodule resolutions of stable Auslander algebras

In this section we specialize to the case where $\mathcal{C} = \operatorname{add}(T)$ for a d-cluster tilting object $T \in \mathcal{B}$ with $d \geq 1$. The evaluation functor $ev_T : M \mapsto M(T)$ gives category equivalences mod-$\mathcal{C} \to$ mod-Λ and mod-$\underline{\mathcal{C}} \to$ mod-Γ, where $\Lambda = \operatorname{End}_{\mathcal{B}}(T)$ and $\Gamma = \underline{\operatorname{End}}_{\mathcal{B}}(T)$. Our Hom-finiteness assumption on $\underline{\mathcal{B}}$ guarantees that Γ is finite-dimensional, although Λ need not be. We also note that Γ may be decomposable as an algebra, and may even have semisimple blocks which we typically want to ignore. As we deal with bimodules, it is convenient to assume that k is perfect (although, it suffices to know that Γ splits over a separable extension of k). Under this assumption, the projective bimodule summands of Γ correspond precisely to semisimple blocks.

We now translate some of our above results (parts (3) and (4) of Theorem 2.5 and (2.11)) to this setting in the corollary below. These statements are also true for $d = 1$ by Theorem 1.1 and Proposition 6.5 of [7].

Corollary 3.1. *Let $T \in \mathcal{B}$ be a d-cluster tilting object with $d \geq 1$ such that $\Omega^d T \cong T$ in $\underline{\mathcal{B}}$, and set $\Lambda = \operatorname{End}_{\mathcal{B}}(T)$ and $\Gamma = \underline{\operatorname{End}}_{\mathcal{B}}(T)$. Then*

1. *$\operatorname{Tor}_i^{\Lambda}(-, \Gamma) = 0$ on mod-Γ for all $i \neq 0, d+1$;*
2. *$\operatorname{Tor}_{d+1}^{\Lambda}(-, \Gamma) \cong \Omega^{d+2}$ as functors on $\underline{\text{mod}}$-Γ.*
3. *$\operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma) \cong \underline{\mathcal{B}}(T, \Omega^d T)$ as (Γ, Γ)-bimodules.*

The assumption that $\Omega^d T \cong T$ implies that $\underline{\mathcal{B}}(T, \Omega^d T)$ is isomorphic to a twisted bimodule ${}_{\sigma}\Gamma_1$ for some k-algebra automorphism of σ of Γ, which corresponds to an isomorphism $\eta : \Omega^d T \xrightarrow{\cong} T$. If $\Omega^d \cong Id$ as functors on $\operatorname{add}(T)$, then $\underline{\mathcal{B}}(T, \Omega^d T) \cong \Gamma$ as bimodules.

Theorem 3.2. *Let $T \in \mathcal{B}$ be a d-cluster tilting object such that $\Omega^d T \cong T$ in $\underline{\mathcal{B}}$, and set $\Lambda = \operatorname{End}_{\mathcal{B}}(T)$ and $\Gamma = \underline{\operatorname{End}}_{\mathcal{B}}(T)$. Then*

1. *$\operatorname{Tor}_{d+1}^{\Lambda}(-, \Gamma) \cong - \otimes_{\Gamma} \operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma)$ as functors on mod-Γ.*
2. *$\Omega_{\Gamma^e}^{d+2}(\Gamma) \cong \operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma) \cong \underline{\mathcal{B}}(T, \Omega^d T)$ as (Γ, Γ)-bimodules (up to projective summands).*

In particular, Γ is self-injective. Moreover, writing $\Gamma = \Gamma_0 \times \Gamma_s$ where Γ_s is the largest semisimple direct factor of Γ, we see that Γ_0 is periodic with period dividing $r(d+2)$ provided $\Omega^d|_{\operatorname{add}(T)}$ has order r as a functor.

Remark. Part (2) and its consequences can be viewed as an extension of Theorem 1.5 in [7]. Notice that we can avoid assuming that Λ has Hochschild dimension $d+1$, even when $d = 1$, since our broader assumptions on $\mathcal{B}$ and T guarantee that Γ is finite-dimensional and self-injective, and we will see that these conditions suffice. In particular, this simplifies certain issues arising in applications of Buchweitz's results (Cf. 1.6, 1.12 in [7]).

Proof. For (1), notice that $\operatorname{Tor}_{d+1}^{\Lambda}(-, \Gamma)$ is an exact functor on mod-Γ as $\operatorname{Tor}_{d}^{\Lambda}(-, \Gamma) = \operatorname{Tor}_{d+2}^{\Lambda}(-, \Gamma) = 0$. Thus $\operatorname{Tor}_{d+1}^{\Lambda}(-, \Gamma) \cong - \otimes_{\Gamma} \operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma)$ by the Eilenberg-Watts theorem. Observe that $\operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma) \cong {}_{\sigma}\Gamma_1$ is a projective Γ-module on either side. Furthermore, since we have an invertible bimodule $\operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma)$ inducing Ω^{d+2} on $\underline{\text{mod}}$-Γ, we see that Ω must be an equivalence and Γ is self-injective.

For (2), let $\cdots P_1 \xrightarrow{f_1} P_0 \longrightarrow \Lambda \to 0$ be a projective resolution of Λ over Λ^e. Applying $- \otimes_{\Lambda^e} \Gamma^e$ yields a complex $Q_{\bullet} := \Gamma \otimes_{\Lambda} P_{\bullet} \otimes_{\Lambda} \Gamma$ of projective Γ^e-modules with homology given by

$$\operatorname{Tor}_*^{\Lambda^e}(\Lambda, \Gamma^e) \cong \operatorname{Tor}_*^{\Lambda}(\Gamma, \Gamma).$$

As Corollary 3.1 tells us that this homology vanishes in all degrees except 0 and $d+1$, the beginning of a projective resolution of Γ over Γ^e has the form

$$0 \to \Omega^{d+2}(\Gamma) \oplus Q \longrightarrow Q_{d+1} \to \cdots \to Q_0 \longrightarrow \Gamma \to 0$$

for some projective bimodule Q. Furthermore, from the definition of Tor we have an epimorphism[1] $\Omega^{d+2}(\Gamma) \oplus Q \to \operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma)$. Let K be the kernel and observe that K is projective on either side since $\operatorname{Tor}_{d+1}^{\Lambda}(\Gamma, \Gamma)$ is. Also observe that by definition $K = \operatorname{im}(1 \otimes f_{d+2} \otimes 1)$ consists of the $(d+1)$-boundaries of $Q_{\bullet}$. We claim that K is a projective (Γ, Γ)-bimodule; since Γ is self-injective it will then follow that the short exact sequence

[1] It is an isomorphism if Λ has Hochschild dimension $d+1$. This holds for instance if $\mathcal{B} =$ mod-A for a finite-dimensional self-injective algebra A, as then Λ is a finite-dimensional algebra of global dimension $d+1$ [16].

$0 \to K \longrightarrow \Omega^{d+2}(\Gamma) \oplus Q \longrightarrow \mathrm{Tor}^{\Lambda}_{d+1}(\Gamma,\Gamma) \to 0$ splits, yielding $\Omega^{d+2}(\Gamma) \cong \mathrm{Tor}^{\Lambda}_{d+1}(\Gamma,\Gamma)$ as bimodules (up to projective summands).

To see that K is projective, we go back a step and apply $\Gamma \otimes_\Lambda -$ to $P_\bullet$ to get a projective (Γ,Λ)-bimodule resolution $\Gamma \otimes_\Lambda P_\bullet$ of ${}_\Gamma\Gamma \otimes_\Lambda \Lambda_\Lambda \cong {}_\Gamma\Gamma_\Lambda$. Set $L = \ker(1 \otimes f_{d+1})$ and let $i : L \to \Gamma \otimes_\Lambda P_{d+1}$ be the natural inclusion. Since $\mathrm{p.dim}(\Gamma_\Lambda) = d+1$, the map i splits as a right Λ-module map. This suffices to deduce that $i \otimes 1 : L \otimes_\Lambda \Gamma \longrightarrow \Gamma \otimes_\Lambda P_{d+1} \otimes_\Lambda \Gamma$ is monic, and it follows that $L \otimes_\Lambda \Gamma \cong \mathrm{im}(1 \otimes f_{d+2} \otimes 1) = K$. For any finitely-presented right Γ-module M, $M \otimes_\Gamma \Gamma \otimes_\Lambda P_\bullet \cong M \otimes_\Lambda P_\bullet$ is a projective resolution of M_Λ. Since $\mathrm{p.dim}\ M_\Lambda \leq d+1$, $1 \otimes i$ splits and $M \otimes_\Gamma L$ is a projective Λ-module. In particular, $M \otimes_\Gamma K \cong M \otimes_\Gamma (L \otimes_\Lambda \Gamma) \cong (M \otimes_\Gamma L) \otimes_\Lambda \Gamma$ is a projective right Γ-module for any M. Since K is projective on either side, Theorem 3.1 of [3] implies that K is a projective bimodule.

For the final statement, we may assume that Γ has no semisimple blocks by working with Γ_0 and an appropriate summand T_0 of T instead. Observe that for any $r \geq 1$, $\Omega^{r(d+2)}(\Gamma) \cong \Omega^{d+2}(\Gamma)^{\otimes r} \cong \underline{\mathcal{B}}(T, \Omega^d T)^{\otimes r}$ up to projective summands by (2) and Corollary 3.1(3). Furthermore $\underline{\mathcal{B}}(T,\Omega^d T)^{\otimes r} \cong \underline{\mathcal{B}}(T, \Omega^{r(d)} T)$ via the map $f_0 \otimes f_1 \otimes \cdots \otimes f_{r-1} \mapsto \Omega^{(r-1)(d)}(f_{r-1}) \cdots \Omega^d(f_1) f_0$, and the latter is isomorphic to $\Gamma = \underline{\mathcal{B}}(T,T)$ as a bimodule if and only if Ω^{rd} is isomorphic to the identity functor on $\mathrm{add}(T)$. □

Many examples of cluster-tilting objects appear inside Calabi-Yau triangulated categories, such as the cluster categories of [5] or categories of the form $\underline{CM}(R)$ for an isolated Gorenstein hypersurface singularity R [8]. Recall that the triangulated category $\underline{\mathcal{B}}$ is (weakly) Calabi-Yau of dimension s if there are natural isomorphisms

$$\underline{\mathcal{B}}(X, Y[s]) \cong D\underline{\mathcal{B}}(Y, X)$$

for all $X, Y \in \mathcal{B}$. In this case, the injective objects in mod-$\underline{\mathcal{B}}$ have the form $D\underline{\mathcal{B}}(X,-) \cong \underline{\mathcal{B}}(-, X[s]) \cong \underline{\mathcal{B}}(-[-s], X)$ for $X \in \mathcal{B}$, which shows that mod-$\underline{\mathcal{B}}$ is a Frobenius category with Nakayama equivalence given by $\nu : F \mapsto F \circ [-s]$. Thus $\underline{\text{mod}}$-$\underline{\mathcal{B}}$ is a Hom-finite triangulated category, and the Auslander-Reiten formula (which applies since $\underline{\mathcal{B}}$ is a dualizing k-variety in the sense of [2]) implies

$$D\underline{\mathrm{Hom}}_{\underline{\mathcal{B}}}(F,G) \cong \mathrm{Ext}^1_{\underline{\mathcal{B}}}(G, D\mathrm{Tr}F) \cong \underline{\mathrm{Hom}}_{\underline{\mathcal{B}}}(G, \Omega\nu F),$$

for all $F, G \in \underline{\text{mod}}$-$\underline{\mathcal{B}}$; that is, $\Omega\nu : F \mapsto \Omega(F \circ [-s])$ is a Serre functor for $\underline{\text{mod}}$-$\underline{\mathcal{B}}$. Moreover, knowledge of the projective resolution for $F \in$ mod-$\underline{\mathcal{B}}$ (from [2], for example) implies that $\Omega^3(F) \cong F \circ [1]$. Hence $\nu \cong \Omega^{-3s}$ on $\underline{\text{mod}}$-$\underline{\mathcal{B}}$, and the Serre functor for $\underline{\text{mod}}$-$\underline{\mathcal{B}}$ satisfies $S = \Omega\nu \cong \Omega^{-(3s-1)}$, showing that $\underline{\text{mod}}$-$\underline{\mathcal{B}}$ is $(3s-1)$-Calabi-Yau when $\underline{\mathcal{B}}$ is s-Calabi-Yau (this has been observed elsewhere: see [20], for instance). This result can in fact be viewed as the $d = 1$ case of a more general statement regarding maximal $(d-1)$-orthogonal subcategories of Calabi-Yau triangulated categories. At the same time, we use Theorem 3.2 to obtain a partial generalization of Proposition 2.1 in [10].

Proposition 3.3. *Let $\mathcal{C}$ be a maximal $(d-1)$-orthogonal subcategory of $\mathcal{B}$ with $\underline{\mathcal{C}}[d] = \underline{\mathcal{C}}$, and assume that $\underline{\mathcal{B}}$ is sd-Calabi-Yau for some integer s.*

1. *$\underline{\text{mod}}$-$\underline{\mathcal{C}}$ is (weakly) Calabi-Yau of dimension $s(d+2)-1$.*
2. *If $\mathcal{C} = \mathrm{add}(T)$ for a d-cluster tilting object $T \in \mathcal{B}$ and $\Gamma = \underline{\mathrm{End}}_{\mathcal{B}}(T)$ has no semisimple blocks, then $\Omega^{-s(d+2)}_{\Gamma^e}(\Gamma) \cong D\Gamma$ as bimodules.*

Proof. (1) As remarked after Lemma 2.1, $\underline{\mathcal{C}}$ is invariant under the Serre functor of $\underline{\mathcal{B}}$. Hence the same argument given above for $\underline{\mathcal{B}}$ shows that the Nakayama equivalence ν on mod-$\underline{\mathcal{C}}$ is given by $F \mapsto F \circ [-sd]$. If $F = \underline{\mathcal{B}}(-, X) \in$ mod-$\underline{\mathcal{C}}$ for $X \in \mathcal{E}_{d-1}$, then $\nu(F) \cong \underline{\mathcal{B}}(-, X[sd]) \cong \Omega_{\underline{\mathcal{C}}}^{-s(d+2)}(F)$ by Theorem 2.5(5). Since $\underline{\mathcal{C}}$ is also a dualizing k-variety [16], the above argument also shows that a Serre functor for $\underline{\text{mod}}$-$\underline{\mathcal{C}}$ is given by $S = \Omega_{\underline{\mathcal{C}}}\nu \cong \Omega_{\underline{\mathcal{C}}}^{1-s(d+2)}$, and the claim follows.

(2) Using Theorem 3.2, we have $\Omega^{-s(d+2)}(\Gamma) \cong \underline{\mathcal{B}}(T, \Omega^{-sd}T) \cong \underline{\mathcal{B}}(T, T[sd]) \cong D\underline{\mathcal{B}}(T,T) \cong D\Gamma$ as bimodules. □

Remark. We point out that the odd requirement that $\underline{\mathcal{B}}$ is sd-Calabi-Yau does not impose an unnecessary restriction in light of the assumption $\underline{\mathcal{C}}[d] = \underline{\mathcal{C}}$. Indeed, if $\underline{\mathcal{B}}$ is n-C.Y. then $\underline{\mathcal{B}}(C, C[n]) \cong D\underline{\mathcal{B}}(C,C) \neq 0$ for any $C \in \mathcal{C}$ implies that $d \mid n$.

4. Examples and concluding remarks

As remarked in the introduction, this work is motivated by the recent discovery of symmetric algebras with $D\mathrm{Tr}$-periodic module categories arising as stable endomorphism rings of 2-cluster tilting objects in the Cohen-Macaulay module categories of 1-dimensional hypersurface singularities [8]. We briefly recall the construction introduced there, as we now know that it provides a powerful tool for producing periodic symmetric algebras of period 4.

Set $S = k[[x, y]]$ and $\mathfrak{m} = (x, y)$. Choose irreducible power series $f_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ for $1 \leq i \leq n$ with $(f_i) \neq (f_j)$ for $i \neq j$, and set $f = f_1 f_2 \cdots f_n$. Then $R = S/(f)$ is an isolated hypersurface singularity of dimension 1, and $T = \oplus_{i=1}^{n} S/(f_1 \cdots f_i)$ is a 2-cluster tilting object in $\mathrm{CM}(R)$. Moreover, Eisenbud's matrix factorization theorem implies that $\Omega^2 \cong Id$ on $\underline{\mathrm{CM}}(R)$, and thus on $\mathrm{add}(T)$ as well. Hence Theorem 3.2 implies that $\Gamma = \underline{\mathrm{End}}_R(T)$ is periodic of period 4. The quiver of Γ (although without relations) is described in Proposition 4.10 of [8]:

$$1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-2 \rightleftarrows n-1$$

with a loop at vertex i if and only if $(f_i, f_{i+1}) \neq \mathfrak{m}$. Furthermore, it is shown that two families of algebras of quaternion type are explicitly realized in this way. These algebras are known to have tame representation type, but starting with a hypersurface R of wild CM-type should produce an algebra Γ of wild type and period 4.

Unfortunately, it is still a challenging problem to find additional examples of maximal $(d-1)$-orthogonal subcategories where our results can be applied. For instance, Erdmann and Holm [12] have shown that maximal $(d-1)$-orthogonal subcategories rarely exist in $\mathcal{B} = \text{mod-}A$ for a self-injective k-algebra A. Specifically, they show that they can only exist if every finite-dimensional A-module has complexity at most 1. Such algebras do exist – periodic algebras, for example – but even here the examples are limited. Known examples of periodic algebras include all self-injective algebra of finite type [9], but any periodic algebra constructed as the stable endomorphism ring of a maximal $(d-1)$-orthogonal subcategory in this context, will again have finite representation type by Lemma 2.3. Still, it would be interesting to see which self-injective algebras of finite representation type are d-cluster tilted in this sense. One could also look for maximal $(d-1)$-orthogonal subcategories of modules over tame and wild periodic algebras, which include the algebras of quaternion type, the preprojective algebras of Dynkin type and the m-fold mesh algebras [14].

Nevertheless, it may still be possible to find interesting examples of d-cluster tilting objects in *subcategories* of stable module categories. In particular, our main results can be applied to a (finite type) maximal $(d-1)$-orthogonal subcategory inside some exact Frobenius subcategory $\mathcal{B}$ of mod-A. Namely, in light of Erdmann and Holm's result, one should take $\mathcal{B}$ to be the full subcategory of mod-A consisting of modules of complexity at most 1, which is an exact subcategory with $\underline{\mathcal{B}}$ a triangulated subcategory of $\underline{\mathrm{mod}}$-A. Even here, however, it is not clear whether one will be able to find a module satisfying the restrictive self-orthogonality and Ext-configuration conditions required of a cluster-tilting object.

Another source of applications can be found in the exciting work of Iyama and Oppermann on *higher preprojective algebras* [18]. If A is a finite-dimensional algebra with gl.dim$A \leq n$ for which mod-A contains an n-cluster-tilting object, then the $(n+1)$-preprojective algebra of A can be defined as $\tilde{A} = T_A \mathrm{Ext}_A^n(DA, A)$, the tensor algebra over A of the bimodule $\mathrm{Ext}_A^n(DA, A)$. Moreover, Iyama and Oppermann show that $\tilde{A}$ can be realized as the endomorphism ring of an n-periodic n-cluster-tilting object in a certain Hom-finite triangulated category (namely, the n-Amiot cluster category $\mathcal{C}_A^n$ associated to A). It follows immediately from Theorem 3.2 that $\tilde{A}$ has at least a quasi-periodic projective resolution over its enveloping algebra. However, it appears a nontrivial problem to determine the order of the n^{th} shift functor $[n]$ on the relevant maximal $(n-1)$-orthogonal subcategory of $\mathcal{C}_A^n$, and thus to determine whether or not this resolution is indeed periodic.

For example, if $n = 1$ and A is a hereditary algebra of finite representation type, then the corresponding 2-preprojective algebra will be the usual preprojective algebra associated to the (Dynkin) quiver of A. Here $\tilde{A}$ is the endomorphism ring of a 1-periodic 1-cluster tilting object T, but has period 6 (with some exceptions in characteristic 2 where the period is 3). This means that for the T in question, one has $T[1] \cong T$ but $-[1] : \mathrm{add}(T) \to \mathrm{add}(T)$ is not isomorphic to the identity functor, although its square $-[2]$ is.

A more interesting example with $n = 2$ can be found in [18], Example 4.18. Here we have a 3-preprojective algebra $\tilde{A}$ for which Ω^{12} fixes each simple module up to isomorphism. Since $\tilde{A}$ is the endomorphism ring of a 2-periodic 2-cluster-tilting object T, with $-[2]|_{\mathrm{add}(T)}$ inducing Ω^4 on $\underline{\mathrm{mod}}$-$\tilde{A}$, we see that the order of $-[2]$ on $\mathrm{add}(T)$ must be a multiple of 3 (if it is finite).

Finally, we point out that Proposition 3.3 applies to all of the $(n+1)$-preprojective algebras $\tilde{A}$, since the relevant n-Amiot cluster category is n-Calabi-Yau by construction. Thus part (2) of the proposition shows that $\Omega^{-n-2}_{\tilde{A}^e}(\tilde{A}) \cong D\tilde{A}$ as bimodules. Since $D\tilde{A} \cong {}_1\tilde{A}_\nu$ for the Nakayama automorphism ν of $\tilde{A}$, we can see that $\tilde{A}$ is periodic if and only if ν has finite order in the group of outer automorphisms of $\tilde{A}$.

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